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Scott power spaces

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Outline

Introduction

An important problem in domain theory is the modelling of non-deterministic features of programming languages and of parallel features treated in a non-deterministic way. If a non-deterministic program runs several times with the same input, it may produce different outputs. To describe this behaviour, powerdomains were introduced by Plotkin [20,21] and Smyth [25] to give denotational semantics to non-deterministic choice in higher-order programming languages.

[20] G. Plotkin, A powerdomain construction, SIAM Journal on Computing. 5 (1976), 452-487.
[21] G. Plotkin, A powerdomain for countable non-determinism, in: Automata, Languages and programming, vol. 140 of Lecture Notes in Computer Science, EATCS, Springer Verlag, 1982 pp. 412-428.
[25] M. Smyth, Powerdomains, J. Computer and Syst. Sci. 16 (1978), 23-36.

Introduction

The three main such powerdomains are the Smyth powerdomain for demonic non-determinism, the Hoare powerdomain for angelic non-determinism, and the Plotkin powerdomain for erratic non-determinism. This viewpoint traditionally stays with the category of dcpos, but is easily and profitably extended to general topological spaces (see, for example, [1,Sections 6.2.3 and 6.2.4] and [23]).

[1] S. Abramsky, A. Jung, Domain theory, in: S. Abramsky, D. Gabbay, T. Maibaum (eds.), Semantic Structures, in: Handbook of Logic in Computer Science, vol.3, Clarendon Press, 1994, pp.1-168.

[23] A. Schalk, Algebras for Generalized Power Constructions, PhD Thesis, Technische Hochschule Darmstadt, 1993.

Introduction

A subset A of a T_0 space X is called *saturated* if A equals the intersection of all open sets containing it (or equivalently, A is an upper set in the specialization order). We shall use $\mathsf{K}(X)$ to denote the set of all nonempty compact saturated subsets of X and endow it with the *Smyth preorder*, that is, for $K_1, K_2 \in \mathsf{K}(X)$, $K_1 \sqsubseteq K_2$ iff $K_2 \subseteq K_1$.

The *upper Vietoris topology* on $\mathsf{K}(X)$ is the topology that has $\{\Box U : U \in \mathcal{O}(X)\}$ as a base, where $\Box U = \{K \in \mathsf{K}(X) : K \subseteq U\}$, and the resulting space is called the *Smyth power space* or *upper space* of X and is denoted by $P_S(X)$. It is easy to verify that the *canonical mapping* $\xi_X : X \longrightarrow P_S(X)$, $x \mapsto \uparrow x$, is a topological embedding.

Introduction

There is another prominent topology one can put on $\mathrm{K}(X)$, namely, the Scott topology. We call the space $\Sigma\mathrm{K}(X) = (\mathrm{K}(X), \sigma(\mathrm{K}(X)))$ the Scott power space of X.

In this talk, we mainly discuss some basic properties of Scott power spaces.

(1) We will prove that the Scott power space $\Sigma\mathrm{K}(X)$ of a well-filtered space X is well-filtered.

(2) It is shown that a T_0 space X is well-filtered iff $\Sigma\mathrm{K}(X)$ is well-filtered and the upper Vietoris topology is coarser than the Scott topology on $\mathrm{K}(X)$.

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(3) A sober space is given for which its Scott power space is not sober.

(4) A few sufficient conditions are given under which a Scott power space is sober.

(5) Some other properties, such as local compactness, first-countability, Rudin property and well-filtered determinedness (WD property for short), of Smyth power spaces and Scott power spaces are also investigated.

Basic concepts and notations

We briefly recall some basic concepts and notations that will be used in the talk.

Let $\mathbb{N}$ denote the poset of all natural numbers with the usual order and ω be the cardinality of $\mathbb{N}$.

For a set X, let $X^{(<\omega)} = \{F \subseteq X : F \text{ is a finite set}\}$ and $X^{(\leqslant\omega)} = \{F \subseteq X : F \text{ is a countable set}\}$.

For a nonempty subset A of P, define $\min(A) = \{u \in A : u \text{ is a minimal element of } A\}$ and $\max(A) = \{v \in A : v \text{ is a maximal element of } A\}$.

Basic concepts and notations

Let $\mathbf{Fin}P = \{\uparrow F : F \in P^{(<\omega)}\}$. The set of all directed sets of P is denoted by $\mathcal{D}(P)$ and the poset of all ideals of P is denoted by $\mathrm{Id}(P)$. Denote the poset of all filters of P by $\mathrm{Filt}(P)$. The poset P is called a *directed complete poset*, or *dcpo* for short, if for any $D \in \mathcal{D}(P)$, $\vee D$ exists in P.

A poset P is called *Noetherian* if it satisfies the *ascending chain condition* (ACC for short): every ascending chain has a greatest member. Clearly, P is Noetherian iff every directed set of P has a largest element (or equivalently, every ideal of P is principal).

Basic concepts and notations

Let X be a topological space X.

(1) X is said to be a *Noetherian space* if every open subset is compact.

(2) X is *locally hypercompact* if for each $x \in X$ and each open neighborhood U of x, there is $\uparrow F \in \mathbf{Fin}X$ such that $x \in \mathrm{int}\uparrow F \subseteq \uparrow F \subseteq U$.

(3) X is called a *c-space* if for each $x \in X$ and each open neighborhood U of x, there is $u \in X$ such that $x \in \mathrm{int}\uparrow u \subseteq \uparrow u \subseteq U$). It is well-known that X is a c-space iff $\mathcal{O}(X)$ is a *completely distributive* lattice.

Basic concepts and notations

A set K of X is called *supercompact* if for any family $\{U_i : i \in I\} \subseteq \mathcal{O}(X)$, $K \subseteq \bigcup_{i \in I} U_i$ implies $K \subseteq U$ for some $i \in I$. It is easy to verify that the supercompact saturated sets of X are exactly the sets $\uparrow x$ with $x \in X$.

Remark 2.1. It is well-known that if a topological space X is locally compact, then it is core-compact (i.e., $\mathcal{O}(X)$ is a continuous lattice).

In [11, Section 7], Hofmann and Lawson gave a second-countable core-compact T_0 space X in which every compact subset of X has empty interior and hence it is not locally compact.

[11] K. Hofmann, J. Lawson, The spectral theory of distributive continuous lattices. Trans. Am. Math.l Soc. 246 (1978) 285-310.

Basic concepts and notations

Lemma 2.2. ([23, Proposition 7.21]) Let X be a T_0 space.

(1) If $\mathcal{K} \in \mathsf{K}(P_S(X))$, then $\bigcup \mathcal{K} \in \mathsf{K}(X)$.

(2) The mapping $\bigcup : P_S(P_S(X)) \longrightarrow P_S(X)$, $\mathcal{K} \mapsto \bigcup \mathcal{K}$, is continuous.

For a T_0 space X, we use $\leq_X$ to denote the *specialization order* of X: $x \leq_X y$ iff $x \in \overline{\{y\}}$).

In the following, when a T_0 space X is considered as a poset, the order always refers to the specialization order if no other explanation.

[23] A. Schalk, Algebras for Generalized Power Constructions, PhD Thesis, Technische Hochschule Darmstadt, 1993.

Basic concepts and notations

Let $\mathcal{O}(X)$ (resp., $\mathcal{C}(X)$) be the set of all open subsets (resp., closed subsets) of X, and let $\mathcal{S}^u(X) = \{\uparrow x : x \in X\}$. Define $\mathcal{S}_c(X) = \{\overline{\{x\}} : x \in X\}$ and $\mathcal{D}_c(X) = \{\overline{D} : D \in \mathcal{D}(X)\}$.

For a T_0 space X and a nonempty subset A of X, A is *irreducible* if for any $\{F_1, F_2\} \subseteq \mathcal{C}(X)$, $A \subseteq F_1 \cup F_2$ implies $A \subseteq F_1$ or $A \subseteq F_2$. Denote by $\mathrm{Irr}(X)$ (resp., $\mathrm{Irr}_c(X)$) the set of all irreducible (resp., irreducible closed) subsets of X.

The *lower Vietoris topology* on $\mathrm{Irr}_c(X)$ is the topology $\{\Diamond U : U \in \mathcal{O}(X)\}$, where $\Diamond U = \{A \in \mathrm{Irr}_c(X) : A \cap U \neq \emptyset\}$. The resulting space, denoted by X^s, with the *canonical mapping* $\eta_X : X \longrightarrow X^s, x \mapsto \overline{\{x\}}$, is the *sobrification* of X. Clearly, $\eta_X : X \longrightarrow X^s$ is a dense topological embedding.

Basic concepts and notations

For a poset P, denote by $\sigma(P)$ the *Scott topology* on P and by $\Sigma P = (P, \sigma(P))$ the *Scott space* of P.

For the chain $2 = \{0, 1\}$ (with the order $1 < 2$), we have $\sigma(2) = \{\emptyset, \{1\}, \{0, 1\}\}$. The space $\Sigma 2$ is well-known under the name of *Sierpiński space*.

The *upper topology* on P, generated by the complements of the principal ideals of P, is denoted by $\upsilon(P)$. The upper sets of P form the (*upper*) *Alexandroff topology* $\alpha(P)$.

d-spaces, well-filtered spaces and sober spaces

A T_0 space X is called a *d-space* (or *monotone convergence space*) if X (with the specialization order) is a dcpo and $\mathcal{O}(X) \subseteq \sigma(X)$ (cf. [6]).

A topological space X is called *sober*, if for any $F \in \mathrm{Irr}_c(X)$, there is a unique point $a \in X$ such that $F = \overline{\{a\}}$ (cf. [6]).

Hausdorff spaces are always sober and sober spaces are always T_0 since $\overline{\{x\}} = \overline{\{y\}}$ always implies $x = y$. The Sierpiński space $\Sigma 2$ is sober but not T_1 and an infinite set with the co-finite topology is T_1 but not sober (see Example 5.4 below).

[6] G. Gierz, K. Hofmann, K. Keimel, J. Lawson, M. Mislove, D. Scott, Continuous Lattices and Domains, Encycl. Math. Appl., vol. 93, Cambridge University Press, 2003.

Proposition 3.1. ([6, 30]) For a T_0 space X, the following conditions are equivalent:

(1) X is a d-space.

(2) $\mathcal{D}_c(X) = \mathcal{S}_c(X)$.

(3) X is a dcpo, and $\overline{D} = \overline{\{\vee D\}}$ for any $D \in \mathcal{D}(X)$.

(4) For any $D \in \mathcal{D}(X)$ and $U \in \mathcal{O}(X)$, $\bigcap_{d \in D} \uparrow d \subseteq U$ implies $\uparrow d \subseteq U$ (i.e., $d \in U$) for some $d \in D$.

[30] X. Xu, C. Shen, X. Xi, D. Zhao, On T_0 spaces determined by well-filtered spaces, Topol. Appl. 282 (2020) 107323.

d-spaces, well-filtered spaces and sober spaces

For the sobriety of the Smyth power spaces, we have the following.

Theorem 3.2. (Heckmann-Keimel-Schalk Theorem) ([10, Theorem 3.13], [23, Lemma 7.20]) For a T_0 space X, the following conditions are equivalent:

(1) X is sober.

(2) For any $\mathcal{A} \in \mathrm{Irr}(P_S(X))$ and $U \in \mathcal{O}(X)$, $\bigcap \mathcal{K} \subseteq U$ implies $K \subseteq U$ for some $K \in \mathcal{A}$.

(3) $P_S(X)$ is sober.

[10] R. Heckmann, K. Keimel, Quasicontinuous domains and the Smyth powerdomain, Electron. Notes Theor. Comput. Sci. 298 (2013) 215-232.

[23] A. Schalk, Algebras for Generalized Power Constructions, PhD Thesis, Technische Hochschule Darmstadt, 1993.

d-spaces, well-filtered spaces and sober spaces

A T_0 space X is called *well-filtered*, if for any filtered family $\mathcal{K} \subseteq \mathrm{K}(X)$ and open set U, $\bigcap \mathcal{K} \subseteq U$ implies $K \subseteq U$ for some $K \in \mathcal{K}$. A well-filtered space will be shortly called a WF space.

We have the following implications (which are irreversible):

$$\text{sobriety} \Rightarrow \text{well-filteredness} \Rightarrow d\text{-space.}$$

Proposition 3.3. ([27, Corollary 3.2]) If a dcpo P endowed with the Lawson topology is compact (in particular, P is a complete lattice), then ΣP is well-filtered.

[27] X. Xi, J. Lawson, On well-filtered spaces and ordered sets, Topol. Appl. 228 (2017) 139-144.

d-spaces, well-filtered spaces and sober spaces

For the well-filteredness of topological spaces, a similar result to Theorem 3.2 was proved in [32] (see also [30]).

Theorem 3.4. ([30, Theorem 5.3], [32, Theorem 4]) For a T_0 space, the following conditions are equivalent:

(1) X is well-filtered.

(2) $P_S(X)$ is a d-space.

(3) $P_S(X)$ is well-filtered.

[30] X. Xu, C. Shen, X. Xi, D. Zhao, On T_0 spaces determined by well-filtered spaces, Topol. Appl. 282 (2020) 107323.

[32] X. Xu, X. Xi, D. Zhao, A complete Heyting algebra whose Scott topology is not sober, Fundam. Math. 252 (2021) 315-323.

By Theorem 3.4 we know that for a well-filtered space X, $\Sigma\mathrm{K}(X)$ is a d-space. Example 6.2 below shows that $\Sigma\mathrm{K}(X)$ is a sober space does not imply that X is a well-filtered space (i.e., $P_S(X)$ is a d-space) in general.

The following example shows that there is a T_0 space X such that $\mathrm{K}(X)$ (with the Smyth order) is a dcpo but X is not well-filtered.

Example 3.5. (Johnstone's dcpo adding a top element) Let $\mathbb{J} = \mathbb{N} \times (\mathbb{N} \cup \{\infty\})$ with ordering defined by $(j, k) \leq (m, n)$ iff $j = m$ and $k \leq n$, or $n = \infty$ and $k \leq m$. $\mathbb{J}$ is a well-known dcpo constructed by Johnstone in [P. Johnstone, Scott is not always sober, in: Continuous Lattices, Lecture Notes in Math., vol. 871, Springer-Verlag, 1981, pp. 282-283].

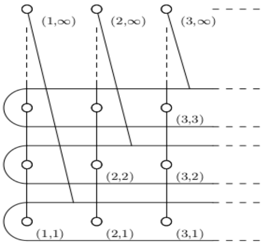

Figure: Johnstone's dcpo $\mathbb{J}$

The set $\mathbb{J}_{max} = \{(n, \infty) : n \in \mathbb{N}\}$ is the set of all maximal elements of $\mathbb{J}$. Adding top $\top$ to $\mathbb{J}$ yields a dcpo $\mathbb{J}_\top = \mathbb{J} \cup \{\top\}$ ($x \leq \top$ for any $x \in \mathbb{J}$). Then $\top$ is the largest element of $\mathbb{J}_\top$ and $\{\top\} \in \sigma(\mathbb{J}_\top)$. The following three conclusions about $\Sigma\mathbb{J}$ are known (see, for example, [17, Example 3.1] and [19, Lemma 3.1]):

(i) $\mathrm{Irr}_c(\Sigma\mathbb{J}) = \{\overline{\{x\}} = \downarrow_{\mathbb{J}} x : x \in \mathbb{J}\} \cup \{\mathbb{J}\}$.

(ii) $\mathsf{K}(\Sigma\mathbb{J}) = (2^{\mathbb{J}_{max}} \setminus \{\emptyset\}) \bigcup \mathbf{Fin}\,\mathbb{J}$.

(iii) $\Sigma\mathbb{J}$ is not well-filtered.

[17] C. Lu, Q. Li, Weak well-filtered spaces and coherence, Topol. Appl. 230 (2017) 373-380.

[19] H. Miao, Q. Li, D. Zhao, On two problems about sobriety of topological spaces, Topol. Appl. 295 (2021) 107667.

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Hence we have

(a) $\mathrm{Irr}_c(\Sigma\mathbb{J}_\top) = \{\overline{\{x\}} = \downarrow_{\mathbb{J}_\top} x : x \in \mathbb{J}_\top\} \cup \{\mathbb{J}\}$ by (i).

(b) $\mathsf{K}(\Sigma\mathbb{J}_\top) = \{\uparrow_{\mathbb{J}_\top} G : G$ is nonempty and $G \subseteq \mathbb{J}_{max} \cup \{\top\}\} \bigcup \mathbf{Fin}\,\mathbb{J}_\top$ by (ii).

(c) $\mathsf{K}(\Sigma\mathbb{J})$ is not a dcpo.

(d) $\mathsf{K}(\Sigma\mathbb{J}_\top)$ is a dcpo.

(e) $\Sigma\mathbb{J}_\top$ is not well-filtered.

Indeed, let $\mathcal{K} = \{\uparrow_{\mathbb{J}_\top}(\mathbb{J}_{max} \setminus F) : F \in (\mathbb{J}_{max})^{(<\omega)}\}$. Then by (b), $\mathcal{K} \subseteq \mathsf{K}(\Sigma\mathbb{J}_\top)$ is filtered and $\bigcap \mathcal{K} = \bigcap_{F \in (\mathbb{J}_{max})^{(<\omega)}} \uparrow_{\mathbb{J}_\top}(\mathbb{J}_{max} \setminus F) = \bigcap_{F \in (\mathbb{J}_{max})^{(<\omega)}}((\mathbb{J}_{max} \setminus F) \cup \{\top\}) = \{\top\} \cup (\mathbb{J}_{max} \setminus \bigcup(\mathbb{J}_{max})^{(<\omega)}) = \{\top\} \in \sigma(\mathbb{J}_\top)$, but there is no $F \in (\mathbb{J}_{max})^{(<\omega)}$ with $\uparrow_{\mathbb{J}_\top}(\mathbb{J}_{max} \setminus F) \subseteq \{\top\}$. Thus $\Sigma\mathbb{J}_\top$ is not well-filtered.

Topological Rudin Lemma, Rudin spaces and well-filtered determined spaces

Rudin's Lemma is a useful tool in non-Hausdorff topology and plays a crucial role in domain theory. In [10] Heckmann and Keimel presented the following topological variant of Rudin's Lemma.

Lemma 4.1. (Topological Rudin Lemma) ([10, Lemma 3.1]) Let X be a topological space and $\mathcal{A}$ an irreducible subset of the Smyth power space $P_S(X)$. Then every closed set $C \subseteq X$ that meets all members of $\mathcal{A}$ contains a minimal irreducible closed subset A that still meets all members of $\mathcal{A}$.

[10] R. Heckmann, K. Keimel, Quasicontinuous domains and the Smyth powerdomain, Electron. Notes Theor. Comput. Sci. 298 (2013) 215-232.

Topological Rudin Lemma, Rudin spaces and well-filtered determined spaces

For a T_0 space X and $\mathcal{K} \subseteq \mathsf{K}(X)$, let $M(\mathcal{K}) = \{A \in \mathcal{C}(X) : K \bigcap A \neq \emptyset \text{ for all } K \in \mathcal{K}\}$ (that is, $\mathcal{K} \subseteq \Diamond A$) and $m(\mathcal{K}) = \{A \in \mathcal{C}(X) : A \text{ is a minimal member of } M(\mathcal{K})\}$.

Definition 4.3. Let X be a T_0 space.

(1) A nonempty subset A of X is said to have the *Rudin property*, if there exists a filtered family $\mathcal{K} \subseteq \mathsf{K}(X)$ such that $\overline{A} \in m(\mathcal{K})$ (that is, $\overline{A}$ is a minimal closed set that intersects all members of $\mathcal{K}$). Let $\mathrm{RD}(X) = \{A \in \mathcal{C}(X) : A \text{ has Rudin property}\}$. The sets in $\mathrm{RD}(X)$ will also be called *Rudin sets*.

(2) X is called a *Rudin space*, RD *space* for short, if $\mathrm{Irr}_c(X) = \mathrm{RD}(X)$, that is, all irreducible closed sets of X are Rudin sets.

Topological Rudin Lemma, Rudin spaces and well-filtered determined spaces

Proposition 4.4. ([24, 30]) Let X be a T_0 space and Y a well-filtered space. If $f : X \longrightarrow Y$ is continuous and $A \subseteq X$ has Rudin property, then there exists a unique $y_A \in X$ such that $\overline{f(A)} = \overline{\{y_A\}}$.

[24] C. Shen, X. Xi, X. Xu, D. Zhao, On well-filtered reflections of T_0 spaces, Topol. Appl. 267 (2019) 106869.

[30] X. Xu, C. Shen, X. Xi, D. Zhao, On T_0 spaces determined by well-filtered spaces, Topol. Appl. 282 (2020) 107323.

Topological Rudin Lemma, Rudin spaces and well-filtered determined spaces

Definition 4.5. ([30]) Let X be a T_0 space.

(1) A subset A of X is called a *well-filtered determined set*, WD *set* for short, if for any continuous mapping $f : X \longrightarrow Y$ to a well-filtered space Y, there exists a unique $y_A \in Y$ such that $\overline{f(A)} = \overline{\{y_A\}}$. Denote by $\mathrm{WD}(X)$ the set of all closed well-filtered determined subsets of X.

(2) X is called a *well-filtered determined* space, WD *space* for short, if all irreducible closed subsets of X are WD sets.

Topological Rudin Lemma, Rudin spaces and well-filtered determined spaces

Proposition 4.6. ([24, 30]) Let X be a T_0 space. Then

$$\mathcal{S}_c(X) \subseteq \mathcal{D}_c(X) \subseteq \mathsf{RD}(X) \subseteq \mathsf{WD}(X) \subseteq \mathsf{Irr}_c(X).$$

Definition 4.7. ([30]) A T_0 space X is called a *directed closure space*, DC *space* for short, if $\mathsf{Irr}_c(X) = \mathcal{D}_c(X)$, that is, for each $A \in \mathsf{Irr}_c(X)$, there exists a directed subset of X such that $A = \overline{D}$.

Corollary 4.8.([30, Corollary 6.3]) Sober $\Rightarrow$ DC $\Rightarrow$ RD $\Rightarrow$ WD.

Proposition 4.9. ([30, Corollary 7.11]) For a T_0 space X, the following conditions are equivalent:

(1) X is well-filtered.

(2) $\mathsf{RD}(X) = \mathcal{S}_c(X)$.

(3) $\mathsf{WD}(X) = \mathcal{S}_c(X)$.

Topological Rudin Lemma, Rudin spaces and well-filtered determined spaces

Theorem 4.10. ([30, Theorem 6.6]) For a T_0 space X, the following conditions are equivalent:

(1) X is sober.

(2) X is a DC d-space.

(3) X is a well-filtered DC space.

(4) X is a well-filtered Rudin space.

(5) X is a well-filtered WD space.

Topological Rudin Lemma, Rudin spaces and well-filtered determined spaces

Proposition 4.11. ([5, Proposition 3.2]) Let X be a locally hypercompact T_0 space and $A \in \mathrm{Irr}(X)$. Then there exists a directed subset $D \subseteq \downarrow A$ such that $\overline{A} = \overline{D}$. Therefore, X is a DC space.

Proposition 4.12. ([30, Theorem 6.10 and Theorem 6.15]) Let X be a T_0 space.

(1) If X is locally compact, then X is a Rudin space.

(2) If X is core-compact, then X is a WD space.

[5] M. Erné, Categories of locally hypercompact spaces and quasicontinuous posets, Appl. Categ. Struct. 26 (2018) 823-854.

[30] X. Xu, C. Shen, X. Xi, D. Zhao, On T_0 spaces determined by well-filtered spaces, Topol. Appl. 282 (2020) 107323.

Topological Rudin Lemma, Rudin spaces and well-filtered determined spaces

It is still not known whether every core-compact T_0 space is a Rudin space (see [34, Question 5.14]).

Question 4.13 For a core-compact T_0 space X, is the Smyth power space $P_S(X)$ a WD space? Is the Scott power space $\Sigma K(X)$ a WD space?

[34] X. Xu, D. Zhao, Some open problems on well-filtered spaces and sober spaces, Topol. Appl. 301(2021)107540.

From Theorem 4.10 and Proposition 4.12 one can immediately get the following result, which was first proved by Lawson, Wu and Xi [16] using a different method.

Corollary 4.14 ([16, 30]) Every core-compact well-filtered space is sober.

[16] J. Lawson, G. Wu, X. Xi, Well-filtered spaces, compactness, and the lower topology, Houst. J. Math. 46 (1) (2020) 283-294.

[30] X. Xu, C. Shen, X. Xi, D. Zhao, On T_0 spaces determined by well-filtered spaces, Topol. Appl. 282 (2020) 107323.

Topological Rudin Lemma, Rudin spaces and well-filtered determined spaces

The following figure shows certain relations among some kinds of spaces.

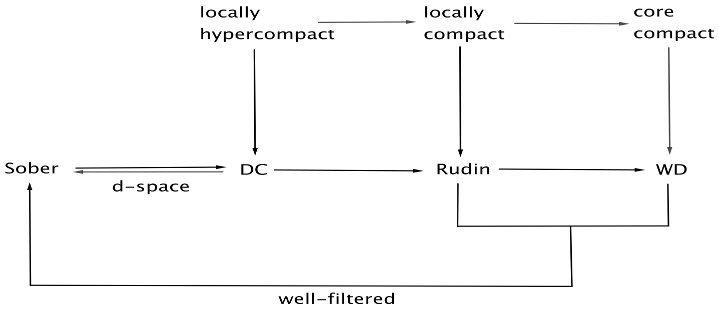

Figure: Certain relations among some kinds of spaces

Topological Rudin Lemma, Rudin spaces and well-filtered determined spaces

Definition 4.15. ([29, Definition 5.1]) Let X be a T_0 space and A a nonempty subset of X.

(a) The set A is said to be an *ω-Rudin set*, if there exists a countable filtered family $\mathcal{K} \subseteq \mathsf{K}(X)$ such that $\overline{A} \in m(\mathcal{K})$. Let $\mathrm{RD}_\omega(X)$ denote the set of all closed ω-Rudin sets of X.

(b) The space X is called *ω-Rudin space*, if $\mathrm{Irr}_c(X) = \mathrm{RD}_\omega(X)$ or, equivalently, all irreducible (closed) subsets of X are ω-Rudin sets.

[29] X. Xu, C. Shen, X. Xi, D. Zhao, First countability, ω-well-filtered spaces and reflections, Topol. Appl. 279 (2020) 107255.

Topological Rudin Lemma, Rudin spaces and well-filtered determined spaces

Definition 4.16. ([29, Definition 3.9]) A T_0 space X is called *ω-well-filtered*, if X is and for any countable filtered family $\{K_n : n < \omega\} \subseteq \mathsf{K}(X)$ and $U \in \mathcal{O}(X)$, it holds that

$$\bigcap_{n<\omega} K_n \subseteq U \;\Rightarrow\; \exists n_0 < \omega, K_{n_0} \subseteq U.$$

An ω-well-filtered space will be shortly called an ω-WF space.

Topological Rudin Lemma, Rudin spaces and well-filtered determined spaces

Definition 4.17. ([29, Definition 5.4]) Let X be a T_0 space and A a nonempty subset of X.

(a) The set A is called an *ω-well-filtered determined set*, ω-WD *set* for short, if for any continuous mapping $f : X \longrightarrow Y$ to an ω-well-filtered space Y, there exists a (unique) $y_A \in Y$ such that $\overline{f(A)} = \overline{\{y_A\}}$. Denote by $\mathrm{WD}_\omega(X)$ the set of all closed ω-well-filtered determined subsets of X.

(b) The space X is called *ω-well-filtered determined*, ω-WD *space* for short, if $\mathrm{Irr}_c(X) = \mathrm{WD}_\omega(X)$ or, equivalently, all irreducible (closed) subsets of X are ω-WD sets.

Topological Rudin Lemma, Rudin spaces and well-filtered determined spaces

By [29, Theorem 5.11], we have the following similar result to Theorem 4.10.

Proposition 4.18. For a T_0 space X, the following conditions are equivalent:

(1) X is sober.

(2) X is an ω-Rudin and ω-WF space.

(3) X is an ω-WD and ω-WF space.

Topological Rudin Lemma, Rudin spaces and well-filtered determined spaces

Theorem 4.19 ([31, Theorem 5.6 and Theorem 6.12]) Let X be a T_0 space.

(1) If the sobrification X^s of X is first-countable, then X is an ω-Rudin space.

(2) If X is first-countable, then X is a WD space.

[31] X. Xu, C. Shen, X. Xi, D. Zhao, First-countability, ω-Rudin spaces and well-filtered determined, Topol. Appl. 300 (2021) 107775.

From Theorem 4.10 and Theorem 4.19 we immediately deduce the following result.

Corollary 4.20. ([29, Theorem 4.2]) Every first-countable well-filtered space is sober.

It is still not known whether a first-countable T_0 space is a Rudin space (see [31, Problem 6.15]).

Since the first-countability is a hereditary property, from Theorem 4.19 we know that if the Smyth power space $P_S(X)$ of a T_0 space X is first-countable, then X is a WD space. Naturally we ask the following question.

Question 4.21. Is a T_0 space with a first-countable Smyth power space a Rudin space?

Topological Rudin Lemma, Rudin spaces and well-filtered determined spaces

In Example 6.1 below a T_0 space X is given for which the Scott power space $\Sigma \mathsf{K}(X)$ is a first-countable sober c-space but X is not a WD space (and hence not a Rudin space).

By Proposition 4.18 and Theorem 4.19, we have the following result.

Corollary 4.22. ([31, Theorem 5.9]) Every ω-well-filtered space with a first-countable sobrification is sober.

[31] X. Xu, C. Shen, X. Xi, D. Zhao, First-countability, ω-Rudin spaces and well-filtered determined, Topol. Appl. 300 (2021) 107775.

Topological Rudin Lemma, Rudin spaces and well-filtered determined spaces

The following example also shows that the first-countability of a T_0 space X does not imply the first-countability of X^s in general.

Example 4.23. Let ω_1 be the first uncountable ordinal number and $P = [0, \omega_1)$. Then

(a) $\mathcal{C}(\Sigma P) = \{\downarrow t : t \in P\} \cup \{\emptyset, P\}$.

(b) ΣP is first-countable and compact (since P has a least element 0).

(c) $(\Sigma P)^s$ is not first-countable.

In fact, it is easy to verify that $(\Sigma P)^s$ is homeomorphic to $\Sigma[0, \omega_1]$. Since sup of a countable family of countable ordinal numbers is still a countable ordinal number, $\Sigma[0, \omega_1]$ has no countable base at the point ω_1.

Topological Rudin Lemma, Rudin spaces and well-filtered determined spaces

(d) $\mathsf{K}(\Sigma P) = \{\uparrow x : x \in P\}$ and ΣP is not an ω-Rudin space.

(e) ΣP is a Rudin space.

(f) P is not a dcpo (note that P is directed and $\vee P$ does not exist). So ΣP is not a d-space, and hence ΣP is neither well-filtered nor sober.

(g) ΣP is ω-well-filtered.

Well-filteredness of Scott power spaces

In this section, we mainly discuss the following two questions:

Question 1. Is the Scott power space $\Sigma \mathrm{K}(X)$ of a d-space X a d-space?

Question 2. Is the Scott power space $\Sigma \mathrm{K}(X)$ of a well-filtered space X well-filtered?

First, Example 5.4 below shows that there is a second-countable Noetherian d-space X for which $\mathrm{K}(X)$ is not a dcpo and hence neither the Smyth power space $P_S(X)$ nor the Scott power space $\Sigma \mathrm{K}(X)$ is a d-space, which gives a negative answer to Question 1.

Well-filteredness of Scott power spaces

Lemma 5.1. ([23, Lemma 7.26]) For a locally compact T_0 space X, the Scott topology is coarser than the upper Vietoris topology on $\mathrm{K}(X)$, that is, $\sigma(\mathrm{K}(X)) \subseteq \mathcal{O}(P_S(X))$.

By Theorem 3.4 and Lemma 5.1, we get the following corollary.

Corollary 5.2. ([23, Lemma 7.26]) If X is a locally compact sober space (equivalently, a locally compact well-filtered space or a core-compact well-filtered space), then the upper Vietoris topology and the Scott topology on $\mathrm{K}(X)$ coincide.

[23] A. Schalk, Algebras for Generalized Power Constructions, PhD Thesis, Technische Hochschule Darmstadt, 1993.

Considering Remark 2.1 and Lemma 5.1, we have the following question.

Question 5.3. For a core-compact T_0 space X, is the Scott topology coarser than the upper Vietoris topology on $\mathrm{K}(X)$?

Now we give a space X which is a second-countable Noetherian d-space, and $\mathrm{K}(X)$ is not a dcpo and hence neither the Smyth power space $P_S(X)$ nor the Scott power space $\Sigma\mathrm{K}(X)$ is a d-space.

Well-filteredness of Scott power spaces

Example 5.4. Let X be a countably infinite set (for example, $X = \mathbb{N}$) and X_{cof} the space equipped with the *co-finite topology* (the empty set and the complements of finite subsets of X are open). Then

(a) $\mathcal{C}(X_{cof}) = \{\emptyset, X\} \cup X^{(<\omega)}$, X_{cof} is T_1 and hence a d-space.

(b) $\mathrm{Irr}_c(X_{cof}) = \{\{x\} : x \in X\} \cup \{X\}$.

(c) $\mathrm{K}(X_{cof}) = 2^X \setminus \{\emptyset\}$.

(d) X_{cof} is second-countable.

(e) X_{cof} is Noetherian and hence locally compact.

(f) X_{cof} is a Rudin space.

(g) $\mathrm{K}(X_{cof})$ is not a dcpo and hence X_{cof} is neither well-filtered nor sober.

(g) The upper Vietoris topology and the Scott topology on $\mathrm{K}(X_{cof})$ agree.

(i) $\Sigma\mathrm{K}(X_{cof})$ is not a d-space and hence it is neither a well-filtered space nor a sober space.

Then we investigate Question 2. First, we have the following important conclusion.

Theorem 5.5. For a well-filtered space X, $\Sigma\mathrm{K}(X)$ is well-filtered.

Well-filteredness of Scott power spaces

Outline of the proof: By Theorem 3.4, $\mathrm{K}(X)$ is a dcpo and $\mathcal{O}(P_S(X)) \subseteq \sigma(\mathrm{K}(X))$ (i.e., $\Box U \in \sigma(\mathrm{K}(X))$ for all $U \in O(X)$). Suppose that $\{\mathcal{K}_d : d \in D\} \subseteq \mathrm{K}(\Sigma\mathrm{K}(X))$ is filtered, $\mathcal{U} \in \sigma(\mathrm{K}(X))$ and $\bigcap_{d \in D} \mathcal{K}_d \subseteq \mathcal{U}$. If $\mathcal{K}_d \not\subseteq \mathcal{U}$ for each $d \in D$, that is, $\mathcal{K}_d \bigcap (\mathrm{K}(X) \setminus \mathcal{U}) \neq \emptyset$, then $\{\mathcal{K}_d : d \in D\} \in \mathrm{Irr}(P_S(\mathrm{K}(\Sigma\mathrm{K}(X))))$ and hence by Lemma 4.1 $\mathrm{K}(X) \setminus \mathcal{U}$ contains a minimal irreducible closed subset $\mathcal{A}$ that still meets all members $\mathcal{K}_d$. For each $d \in D$, let $K_d = \bigcup \uparrow_{\mathrm{K}(X)}(\mathcal{K}_d \cap \mathcal{A})$.

Claim 1: For each $d \in D$, $K_d \in \mathrm{K}(X)$ and $K_d \in \mathcal{A}$.

Claim 2: $\{K_d : d \in D\} \subseteq \mathrm{K}(X)$ is filtered (by Claim 1 and the filteredness of $\{\mathcal{K}_d : d \in D\}$).

Claim 3: $K = \bigcap_{d \in D} K_d \in \mathrm{K}(X)$ and $K \in \mathcal{A}$.

Well-filteredness of Scott power spaces

Claim 4: For each $k \in K$, $\mathcal{A} \subseteq \Diamond_{\mathsf{K}(X)}\overline{\{k\}}$.

Claim 5: $A = \downarrow_{\mathsf{K}(X)} K$.

Claim 6: $K \in \bigcap_{d \in D} \mathcal{K}_d$.

Therefore, there is $d_0 \in D$ such that $\mathcal{K}_{d_0} \subseteq \mathcal{U}$, proving that $\Sigma\mathsf{K}(X)$ is well-filtered.

Example 6.2 below shows that unlike Smyth power spaces (see Theorem 3.4), the converse of Theorem 5.5 does not hold.

From Theorem 4.15 and Theorem 5.5 we deduce the following result.

Corollary 5.6. For a well-filtered space X, the following two conditions are equivalent:

(1) $\Sigma \mathrm{K}(X)$ is core-compact.

(2) $\Sigma \mathrm{K}(X)$ is locally compact.

Well-filteredness of Scott power spaces

By Theorem 4.10, Theorem 4.15 and Theorem 5.5, we have the following two corollaries.

Corollary 5.7. For a well-filtered space X, the following three conditions are equivalent:

(1) $\Sigma \mathsf{K}(X)$ is sober.

(2) $\Sigma \mathsf{K}(X)$ is Rudin.

(3) $\Sigma \mathsf{K}(X)$ is a WD space.

Corollary 5.8. Let X be a well-filtered space.

(1) If $\mathcal{K} \in \mathsf{K}(\Sigma \mathsf{K}(X))$, then $\bigcup \mathcal{K} \in \mathsf{K}(X)$.

(2) The mapping $\bigcup : \Sigma \mathsf{K}(\Sigma \mathsf{K}(X)) \longrightarrow \Sigma \mathsf{K}(X)$, $\mathcal{K} \mapsto \bigcup \mathcal{K}$, is continuous.

Proposition 5.9. Let X be a T_0 space. If the upper Vietoris topology is coarser than the Scott topology on $\mathsf{K}(X)$ (that is, $\mathcal{O}(P_S(X)) \subseteq \sigma(\mathsf{K}(X))$), and $\Sigma\mathsf{K}(X)$ is well-filtered, then X is well-filtered.

Example 6.2 below shows that when X lacks the condition of $\mathcal{O}(P_S(X)) \subseteq \sigma(\mathsf{K}(X))$, Proposition 5.9 may not hold.

Well-filteredness of Scott power spaces

Corollary 5.10. For a T_0 space X, the following conditions are equivalent:

(1) X is well-filtered.

(2) The upper Vietoris topology is coarser than the Scott topology on $\mathrm{K}(X)$, and $\Sigma \mathrm{K}(X)$ is well-filtered.

(3) The upper Vietoris topology is coarser than the Scott topology on $\mathrm{K}(X)$, and $\Sigma \mathrm{K}(X)$ is a d-space.

(4) $\mathrm{K}(X)$ is a dcpo, and the upper Vietoris topology is coarser than the Scott topology on $\mathrm{K}(X)$.

Non-sobriety of Scott power space of a sober space

In this section, we investigate the following question:

Question 3. Is the Scott power space $\Sigma \mathrm{K}(X)$ of a sober space X sober?

First, the following example shows that there is a well-filtered space X for which its Scott power space $\Sigma \mathrm{K}(X)$ is a first-countable sober c-space, but X is not sober. Hence, by Corollary 4.14 and Corollary 4.20, X is neither core-compact nor first-countable. So the sobriety (resp., first-countability, local compactness) of the Scott power space of a T_0 space X does not imply the sobriety (resp., first-countability, local compactness) of X in general.

Non-sobriety of Scott power space of a sober space

Example 6.1. Let X be an uncountably infinite set and X_{coc} the space equipped with *the co-countable topology* (the empty set and the complements of countable subsets of X are open). Then

(a) $\mathcal{C}(X_{coc}) = \{\emptyset, X\} \bigcup X^{(\leqslant\omega)}$, X_{coc} is T_1 and hence a d-space, and the specialization order on X_{coc} is the discrete order.

(b) Neither X_{coc} nor $P_S(X_{coc})$ is first-countable.

(c) $\mathrm{Irr}_c(X_{coc}) = \{\overline{\{x\}} : x \in X\} \cup \{X\} = \{\{x\} : x \in X\} \cup \{X\}$. Therefore, X_{coc} is not sober.

(d) $\mathrm{K}(X_{coc}) = X^{(<\omega)} \setminus \{\emptyset\}$ and $\mathrm{int}\, K = \emptyset$ for all $K \in \mathrm{K}(X_{coc})$, and hence X_{coc} is not locally compact.

(e) X_{coc} is well-filtered and not core-compact.

(f) $\sigma(\mathrm{K}(X_{coc})) = \alpha(\mathrm{K}(X_{coc}))$ and hence $\Sigma\mathrm{K}(X_{coc})$ is first-countable.

Non-sobriety of Scott power space of a sober space

(g) The upper Vietoris topology and the Scott topology on $\mathrm{K}(X_{coc})$ do not agree.

(h) The Scott power space $\Sigma\mathrm{K}(X_{coc})$ is a sober c-space. So it is a Rudin space and a WD space.

(i) X_{coc} is neither a Rudin space nor a WD space.

(j) The Smyth power space $P_S(X_{coc})$ is well-filtered but non-sober. Hence it is neither a Rudin space nor a WD space.

(k) $P_S(X_{coc})$ is not core-compact.

Non-sobriety of Scott power space of a sober space

The following example shows there is even a second-countable Noetherian T_0 space X such that the Scott power space $\Sigma\mathrm{K}(X)$ is a second-countable sober space but X is not well-filtered (and hence not sober).

Example 6.2. Let $P = \mathbb{N} \cup \{\infty\}$ and define an order on P by $x \leq_P y$ iff $x = y$ or $x \in \mathbb{N}$ and $y = \infty$.

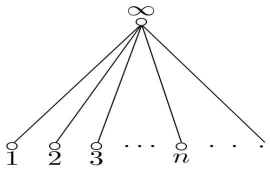

Figure: The poset P

Non-sobriety of Scott power space of a sober space

Let $\tau = \{(\mathbb{N} \setminus F) \cup \{\infty\} : F \in \mathbb{N}^{(<\omega)}\} \cup \{\emptyset, P\} \cup \{\{\infty\}\}$. It is straightforward to verify that τ is a T_0 topology on P and the specialization order of (P, τ) agrees with the original order on P. Now we have

(a) $\mathcal{C}((P, \tau)) = \mathbb{N}^{(<\omega)} \cup \{\emptyset, P\} \cup \{\mathbb{N}\}$.

(b) $\mathrm{Irr}_c((P, \tau)) = \{\overline{\{n\}} = \{n\} : n \in \mathbb{N}\} \cup \{\overline{\{\infty\}} = P\} \cup \{\mathbb{N}\}$ and hence (P, τ) is not sober.

(c) $\mathrm{K}((P, \tau)) = \{A \cup \{\infty\} : A \subseteq \mathbb{N}\}$.

(d) (P, τ) is not well-filtered.

(e) (P, τ) is Noetherian and second-countable and hence it is a Rudin space.

(f) $\Sigma \mathrm{K}((P, \tau))$ is a second-countable sober space.

Non-sobriety of Scott power space of a sober space

(g) $P_S((P, \tau))$ is second-countable by Proposition 7.7.

(h) $\sigma(\mathsf{K}((P, \tau))) \subseteq \mathcal{O}(P_S((P, \tau)))$ but $\mathcal{O}(P_S((P, \tau))) \nsubseteq \sigma(\mathsf{K}((P, \tau)))$.

In the following we will construct a sober space X for which its Scott power space is non-sober (see Theorem 6.7 below).

Non-sobriety of Scott power space of a sober space

Let $\mathcal{L} = \mathbb{N} \times \mathbb{N} \times (\mathbb{N} \cup \{\infty\})$, where $\mathbb{N}$ is the set of natural numbers with the usual order. Define an order $\leq$ on $\mathcal{L}$ as follows:

$(i_1, j_1, k_1) \leq (i_2, j_2, k_2)$ if and only if:

(1) $i_1 = i_2, j_1 = j_2, k_1 \leq k_2 \leq \infty$; or

(2) $i_2 = i_1 + 1, k_1 \leq j_2, k_2 = \infty$.

$\mathcal{L}$ is a known dcpo constructed by Jia in [13, Example 2.6.1]. It can be easily depicted as in the following figure taken from [13].

[13] X. Jia, Meet-Continuity and Locally Compact Sober Dcpos, PhD thesis, University of Birmingham, 2018.

Non-sobriety of Scott power space of a sober space

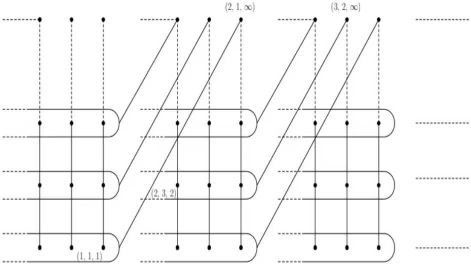

Figure: A non-sober well-filtered dcpo $\mathcal{L}$

Non-sobriety of Scott power space of a sober space

Lemma 6.3. $\mathrm{Irr}_c(\Sigma\mathcal{L}) = \{\overline{\{x\}} = \downarrow x : x \in \mathcal{L}\} \cup \{\mathcal{L}\}$.

Proposition 6.4. ([13, Example 2.6.1]) $\Sigma\mathcal{L}$ is well-filtered but non-sober.

Definition 6.5. Let X be a T_0 space for which X is irreducible (i.e., $X \in \mathrm{Irr}_c(X)$). Choose a point $\top$ such that $\top \notin X$. Then $(\mathcal{C}(X) \setminus \{X\}) \cup \{X \cup \{\top\}\}$ (as the set of all closed sets) is a topology on $X \cup \{\top\}$. The resulting space is denoted by $X_\top$. Define a mapping $\zeta_X : X \to X_\top$ by $\zeta_X(x) = x$ for each $x \in X$. It is easy to verify that η_X is a topological embedding.

Non-sobriety of Scott power space of a sober space

Lemma 6.6. $\langle(\Sigma\mathcal{L})_\top, \zeta_\mathcal{L}\rangle$ is a sobrification of $\Sigma\mathcal{L}$, where $\zeta_\mathcal{L} : \Sigma\mathcal{L} \to (\Sigma\mathcal{L})_\top$ is defined by $\zeta_\mathcal{L}(x) = x$ for each $x \in \mathcal{L}$.

Theorem 6.7. The Scott power space $\Sigma\mathrm{K}((\Sigma\mathcal{L})_\top)$ of the sober space $(\Sigma\mathcal{L})_\top$ is non-sober.

By Theorem 3.2, Theorem 3.4, Theorem 6.7 and Corollary 7.6 below, we naturally pose the following question.

Question 6.8 For a T_2 space X, is the Scott power space $\Sigma\mathrm{K}(X)$ sober?

Local compactness, first-countability and sobriety of Smyth power spaces and Scott power spaces

By Corollary 4.14, Corollary 4.20 and Theorem 5.5, we get the following.

Corollary 7.1. If X is a well-filtered space for which the Scott power space $\Sigma \mathrm{K}(X)$ is first-countable or core-compact (especially, locally compact), then $\Sigma \mathrm{K}(X)$ is sober.

For the local compactness of Smyth power spaces, we have the following.

Local compactness, first-countability and sobriety of Smyth power spaces and Scott power spaces

Lemma 7.2. ([18, Theorem 3.1]) For a T_0 space, the following conditions are equivalent:

(1) X is locally compact.

(2) $P_S(X)$ is core-compact.

(3) $P_S(X)$ is locally compact.

(4) $P_S(X)$ is locally hypercompact.

(5) $P_S(X)$ is a c-space.

[18] Z. Lyu, Y. Chen, X. Jia. Core-compactness, consonance and the Smyth powerspaces. Topol. Appl. 312 (2022) 108066.

Local compactness, first-countability and sobriety of Smyth power spaces and Scott power spaces

The following corollary follows directly from Proposition 4.11 and Lemma 7.2.

Corollary 7.3. For a locally compact T_0 space X, the Smyth power space $P_S(X)$ is a DC space.

Concerning the Scott power space of a locally compact T_0 space, we have the following question.

Question 7.4. For a locally compact T_0 space X, is the Scott power space $\Sigma \mathrm{K}(X)$ a Rudin space or a WD space?

Local compactness, first-countability and sobriety of Smyth power spaces and Scott power spaces

Proposition 7.5. Let X be a locally compact sober space. Then

(1) the Scott power space of X and the Symth power space of X coincide, that is, $\Sigma \mathrm{K}(X) = P_S(X)$.

(2) $\Sigma \mathrm{K}(X)$ is a sober c-space.

Corollary 7.6. If X is a locally compact T_2 (especially, a compact T_2) space, then

(1) $\Sigma \mathrm{K}(X) = P_S(X)$.

(2) $\Sigma \mathrm{K}(X)$ is a sober c-space.

Local compactness, first-countability and sobriety of Smyth power spaces and Scott power spaces

For the Smyth power spaces and sobrifications of T_0 spaces, we have the following conclusion.

Proposition 7.7. ([2, 31, 33]) For a T_0 space, the following conditions are equivalent:

(1) X is second-countable.

(2) $P_S(X)$ is second-countable.

(3) X^s is second-countable.

[2] M. Brecht, T. Kawai, On the commutativity of the powerspace constructions, Logic Mothods in Compt. Sci. 15(3) (2019) 13:1-13:25.

[31] X. Xu, C. Shen, X. Xi, D. Zhao, First-countability, ω-Rudin spaces and well-filtered determined, Topol. Appl. 300 (2021) 107775.

[33] X. Xu and Z. Yang, Coincidence of the upper Vietoris topology and the Scott topology, Topol. Appl. 288 (2021) 107480

Local compactness, first-countability and sobriety of Smyth power spaces and Scott power spaces

As first-countability is a hereditary property, we have the following.

Proposition 7.8. Let X be a T_0 space. If X^s is first-countable or $P_S(X)$ is first-countable, then X is first-countable.

Example 6.1 shows that unlike the Smyth power space, the first-countability of the Scott power space of a T_0 space X does not imply the first-countability of X in general.

The converse of Proposition 7.8 does not hold in general, as shown in Example 4.23 and the following example. It also shows that even for a compact Hausdorff first-countable space X, the Scott power space of X and the Smyth power space of X may not be first-countable.

Local compactness, first-countability and sobriety of Smyth power spaces and Scott power spaces

Example 7.9. Consider in the plane $\mathbb{R}^2$ two concentric circles $C_i = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = i\}$, where $i = 1, 2$, and their union $X = C_1 \cup C_2$; the projection of C_1 onto C_2 from the point $(0, 0)$ is denoted by p. On the set X we generate a topology by defining a neighbourhood system $\{B(z) : z \in X\}$ as follows: $B(z) = \{z\}$ for $z \in C_2$ and $B(z) = \{U_j(z) : j \in \mathbb{N}\}$ for $z \in C_1$, where $U_j = V_j \bigcup p(V_j \setminus \{z\})$ and V_j is the arc of C_1 with center at z and of length $1/j$. The space X is called the *Alexandroff double circle*. The following conclusions about X are known (see, for example, [3, Example 3.1.26]).

[3] R. Engelking, General Topology, Polish Scientific Publishers, Warzawa, 1989.

Local compactness, first-countability and sobriety of Smyth power spaces and Scott power spaces

(a) X is Hausdorff and first-countable.

(b) X is compact and locally compact.

(c) X is not separable, and hence not second-countable.

(d) C_1 is a compact subspace of X.

(e) C_2 is a discrete subspace of X.

We can prove that there is no countable base at C_1 in $P_S(X)$. Thus $P_S(X)$ is not first-countable. For details, see [33, Example 4.4]. By Corollary 7.6, $\Sigma \mathrm{K}(X) = P_S(X)$, whence the Scott power space of X is not first-countable.

[33] X. Xu and Z. Yang, Coincidence of the upper Vietoris topology and the Scott topology, Topol. Appl. 288 (2021) 107480

Local compactness, first-countability and sobriety of Smyth power spaces and Scott power spaces

Proposition 7.10. ([33, Proposition 4.5]) Let X be a first-countable T_0 space. If $\min(K)$ is countable for any $K \in \mathrm{K}(X)$, then $P_S(X)$ is first-countable.

Proposition 7.11. For a metric space (X, d), $P_S((X, d))$ is first-countable.

Local compactness, first-countability and sobriety of Smyth power spaces and Scott power spaces

For a countable T_0 space X, it is easy to see that X is second-countable iff X is first-countable. Therefore, by Proposition 7.7 and Proposition 7.8, we have the following.

Corollary 7.12. For a countable T_0 space X, the following conditions are equivalent:

(1) X is first-countable.

(2) X is second-countable.

(3) X^s is first-countable.

(4) X^s is second-countable.

(5) $P_S(X)$ is first-countable.

(6) $P_S(X)$ is second-countable.

Local compactness, first-countability and sobriety of Smyth power spaces and Scott power spaces

It is worth noting that the Scott topology on a countable complete lattice may not be first-countable, see [29, Example 4.8].

By Proposition 4.18, Theorem 4.19, Proposition 7.7 and Corollary 7.12, we deduce the following two results.

Corollary 7.13. ([31, Corollary 5.7 and Corollary 5.8]) Every second-countable (especially, countable first-countable) T_0 space is an ω-Rudin space.

[29] X. Xu, C. Shen, X. Xi, D. Zhao, First countability, ω-well-filtered spaces and reflections, Topol. Appl. 279 (2020) 107255.

[31] X. Xu, C. Shen, X. Xi, D. Zhao, First-countability, ω-Rudin spaces and well-filtered determined, Topol. Appl. 300 (2021) 107775.

Corollary 7.14. Every second-countable (especially, countable first-countable) ω-well-filtered space is sober.

For a T_0 space X with a first-countable Smyth power space, we have a similar result to Lemma 5.1.

Lemma 7.15. Let X be a T_0 space for which the Smyth power space $P_S(X)$ is first-countable. Then the Scott topology is coarser than the upper Vietoris topology on $\mathrm{K}(X)$.

Local compactness, first-countability and sobriety of Smyth power spaces and Scott power spaces

The following conclusion is straightforward from Theorem 3.4, Corollary 4.20 and Lemma 7.15.

Corollary 7.16. ([33, Theorem 5.7]) Let X be a well-filtered space for which the Smyth power space $P_S(X)$ is first-countable. Then

(1) the upper Vietoris topology agrees with the Scott topology on $\mathrm{K}(X)$.

(2) the Scott power space $\Sigma\mathrm{K}(X)$ is a first-countable sober space.

[33] X. Xu and Z. Yang, Coincidence of the upper Vietoris topology and the Scott topology, Topol. Appl. 288 (2021) 107480.

By Proposition 7.10 and Corollary 7.16, we obtain the following.

Corollary 7.17 ([33, Corollary 5.10]) Let X be a first-countable well-filtered space X in which all compact subsets are countable (especially, $|X| \leq \omega$). Then

(1) the upper Vietoris topology agrees with the Scott topology on $\mathsf{K}(X)$.

(2) the Scott power space $\Sigma\mathsf{K}(X)$ is a first-countable sober space.

By Example 6.1, Lemma 7.15 and Corollary 7.16, we naturally pose the following four questions.

Question 7.18 For a first-countable T_0 space X, is the Scott topology coarser than the upper Vietoris topology on $\mathsf{K}(X)$?

Question 7.19 For a first-countable well-filtered (or equivalently, a first-countable sober) space X, does the upper Vietoris topology and the Scott topology on $\mathsf{K}(X)$ coincide?

Question 7.20 For a first-countable T_2 space X, does the upper Vietoris topology and the Scott topology on $\mathrm{K}(X)$ coincide?

Question 7.21 Is the Scott power space of a first-countable well-filtered (or equivalently, a first-countable sober) space sober?

Local compactness, first-countability and sobriety of Smyth power spaces and Scott power spaces

By Proposition 3.4, Proposition 7.5, Proposition 7.11 and Corollary 7.16, we get the following conclusion.

Corollary 7.22 Let (X, d) be a metric space. Then

(1) the upper Vietoris topology agrees with the Scott topology on $\mathsf{K}((X, d))$.

(2) the Scott power space $\Sigma\mathsf{K}((X, d))$ is a first-countable sober space.

If, in addition, (X, d) is locally compact (especially, compact), then

(3) $\mathsf{K}((X, d))$ is a continuous semilattice.

(4) the Scott power space $\Sigma\mathsf{K}((X, d))$ is a c-space.

Local compactness, first-countability and sobriety of Smyth power spaces and Scott power spaces

The following two conclusions follow directly from Proposition 7.7, Lemma 7.15 and Corollary 7.16.

Corollary 7.23 Let X be a second-countable T_0 space. Then the Scott topology is coarser than the upper Vietoris topology on $\mathrm{K}(X)$.

Corollary 7.24 Let X be a second-countable well-filtered space (or equivalently, a second-countable sober space). Then

(1) the Scott topology agrees with the upper Vietoris topology on $\mathrm{K}(X)$.

(2) the Scott power space of X is a second-countable sober space.

Local compactness, first-countability and sobriety of Smyth power spaces and Scott power spaces

The following example shows that there is a countable Hausdorff space X for which the Scott power space $\Sigma \mathrm{K}(X)$ is second-countable but X is not first-countable (and hence $P_S(X)$ is not first-countable).

Example 7.25. Let p be a point in $\beta(\mathbb{N}) \setminus \mathbb{N}$, where $\beta(\mathbb{N})$ is the Stone-Cěch compactification of the discrete space of natural numbers, and consider on $X = \mathbb{N} \cup \{p\}$ the induced topology (cf. [6, Example II-1.25]). Then

[6] G. Gierz, K. Hofmann, K. Keimel, J. Lawson, M. Mislove, D. Scott, Continuous Lattices and Domains, Encycl. Math. Appl., vol. 93, Cambridge University Press, 2003.

Local compactness, first-countability and sobriety of Smyth power spaces and Scott power spaces

(a) $|X| = \omega$ and X is a non-discrete Hausdorff space and hence a sober space.

(b) $\mathsf{K}(X) = X^{(<\omega)} \setminus \{\emptyset\}$ and $\mathrm{int}K = \emptyset$ for each $K \in \mathsf{K}(X)$. So X is not locally compact.

(c) $\mathsf{K}(X)$ is a Noetherian poset and $|\mathsf{K}(X)| = \omega$. Hence the Scott power space $\Sigma\mathsf{K}(X)$ is a second-countable sober c-space.

(d) the upper Vietoris topology and the Scott topology on $\mathsf{K}(X)$ does not coincide, or more precisely, $\sigma(\mathsf{K}(X)) \not\subseteq \mathcal{O}(P_S(X))$.

(e) Neither X nor $P_S(X)$ is first-countable.

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The above example also shows that if the Smyth power space is replaced with the Scott power space in the conditions of Lemma 7.15 and Corollary 7.16, the analogous results to Lemma 7.15 and Corollary 7.16 do not hold.

By Proposition 7.7, Lemma 7.15, Corollary 7.16 and Corollary 7.24, we pose the following question.

Question 7.26 For a second-countable T_0 space X, is the Scott power space of X second-countable?

Rudin property and well-filtered determinedness of Smyth power sapces and Scott power spaces

Proposition 8.1. ([30, Theorem 7.21]) Let X be a T_0 space. If $P_S(X)$ is a WD space, then X is a WD space.

It is still not known whether the converse of Proposition 8.1 holds (see [30, Question 8.6]).

By Theorem 3.4 and Theorem 4.10, we have the following.

Proposition 8.2. Let X be a well-filtered space. Then the following conditions are equivalent:

(1) X is a Rudin space.

(2) X is a WD space.

(3) $P_S(X)$ is a Rudin space.

(4) $P_S(X)$ is a WD space.

Rudin property and well-filtered determinedness of Smyth power sapces and Scott power spaces

For the Rudin property, we have a similar result to Proposition 8.1.

Theorem 8.3. Let X be a T_0 space. If $P_S(X)$ is a Rudin space, then X is a Rudin space.

Question 8.4. Is the Smyth power space $P_S(X)$ of a Rudin space X still a Rudin space?

Definition 8.5. A T_0 space X is said to have *property S* if for each $A \in \mathrm{Irr}_c(X)$, $\{\uparrow a : a \in A\} \in \mathrm{Irr}(\Sigma \mathrm{K}(X))$ or $\Diamond A \in \mathrm{Irr}_c(\Sigma \mathrm{K}(X))$. A poset P is said to have property S if ΣP has property S.

Rudin property and well-filtered determinedness of Smyth power sapces and Scott power spaces

Remark 8.6. Let X be a T_0 space and $A \in \mathrm{Irr}_c(X)$.

(1) Since $\xi_X : X \longrightarrow P_S(\mathrm{K}(X)), x \mapsto \uparrow x$, is continuous, $\{\uparrow a : a \in A\} \in \mathrm{Irr}(P_S(X))$ and $\mathrm{cl}_{\mathcal{O}(P_S(X))}\{\uparrow a : a \in A\} = \Diamond A \in \mathrm{Irr}_c(P_S(X))$.

(2) If $\xi_X^\sigma : X \longrightarrow \Sigma\mathrm{K}(X), x \mapsto \uparrow x$, is continuous, then X has property S.

(3) If $\sigma(\mathrm{K}(X)) \subseteq \mathcal{O}(P_S(X))$, then $\xi_X^\sigma : X \longrightarrow \Sigma\mathrm{K}(X)$ is continuous, and hence X has property S.

(4) For a poset P, the mapping $\xi_P^\sigma : \Sigma P \longrightarrow \Sigma\mathrm{K}(\Sigma P), x \mapsto \uparrow x$, is continuous. Therefore, P has property S.

Proposition 8.7. Suppose that a T_0 space X has property S and $\mathcal{O}(P_S(X)) \subseteq \sigma(\mathsf{K}(X))$. If $\Sigma\mathsf{K}(X)$ is a Rudin space, then X is a Rudin space.

Corollary 8.8. Suppose that X is a well-filtered space with property S. If $\Sigma\mathsf{K}(X)$ is a WD space (especially, a Rudin space), then both $\Sigma\mathsf{K}(X)$ and X are sober.

By Remark 8.6 and Corollary 8.8, we have the following corollary.

Corollary 8.9. Let X be a well-filtered space. If $\xi_X^\sigma : X \longrightarrow \Sigma\mathsf{K}(X)$ is continuous and $\Sigma\mathsf{K}(X)$ is a WD space (especially, a Rudin space), then both $\Sigma\mathsf{K}(X)$ and X are sober.

Rudin property and well-filtered determinedness of Smyth power sapces and Scott power spaces

As an immediate corollary of Corollary 8.9 we get one the main results of [32].

Corollary 8.10. ([32, Theorem 2]) Suppose that X is a well-filtered space and $\xi_X^\sigma : X \longrightarrow \Sigma \mathsf{K}(X)$ is continuous. If $\Sigma \mathsf{K}(X)$ is sober, then X is sober. Therefore, if X is non-sober, then its Scott power space $\Sigma \mathsf{K}(X)$ is non-sober.

Example 6.1 shows that when X lacks the property S or the continuity of $\xi_X^\sigma : X \longrightarrow \Sigma \mathsf{K}(X)$, Proposition 8.7, Corollary 8.8, Corollary 8.9 and Corollary 8.10 may not hold.

[32] X. Xu, X. Xi, D. Zhao, A complete Heyting algebra whose Scott topology is not sober, Fundam. Math. 252 (2021) 315-323.

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By Remark 8.6, Proposition 8.7, Corollary 8.8 and Corollary 8.10, we deduce the following three corollaries.

Corollary 8.11. Let P be a dcpo. If $\mathcal{O}(P_S(\Sigma P)) \subseteq \sigma(\mathsf{K}(\Sigma P))$ and $\Sigma\mathsf{K}(\Sigma P)$ is a Rudin space, then ΣP is a Rudin dcpo.

Corollary 8.12. Let P be a dcpo. If $\Sigma\mathsf{K}(\Sigma P)$ is a WD space (especially, a Rudin space), then both $\Sigma\mathsf{K}(\Sigma P)$ and ΣP are sober.

Corollary 8.13. Let P be a dcpo for which ΣP is well-filtered. If $\Sigma\mathsf{K}(\Sigma P)$ is sober, then ΣP is sober. Therefore, if ΣP is non-sober, then $\Sigma\mathsf{K}(\Sigma P)$ is non-sober.

Rudin property and well-filtered determinedness of Smyth power sapces and Scott power spaces

Example 8.14. Let L be the Isbell's lattice constructed in [12]. Then

(a) ΣL is non-sober.

(b) ΣL is well-filtered by Proposition 3.3; whence ΣL is neither a Rudin space nor a WD space by (a) and Corollary 5.18.

(c) $\Sigma \mathrm{K}(\Sigma L)$ is well-filtered by Theorem 5.5.

(d) $\mathrm{K}(\Sigma L)$ is a spatial frame (see [32, Lemma 1]).

(e) $\Sigma \mathrm{K}(\Sigma L)$ is not sober by (a) and Corollary 8.13.

(f) $\Sigma \mathrm{K}(\Sigma L)$ is neither a Rudin space nor a WD space by (a)(b) and Corollary 8.12.

[12] J. Isbell, Completion of a construction of Johnstone, Proc. Amer. Math. Soci. 85 (1982) 333-334.
[32] X. Xu, X. Xi, D. Zhao, A complete Heyting algebra whose Scott topology is not sober, Fundam. Math. 252 (2021) 315-323.

Rudin property and well-filtered determinedness of Smyth power sapces and Scott power spaces

Proposition 8.15 Suppose that X is a T_0 space for which $\sigma(\mathsf{K}(X)) \subseteq \mathcal{O}(P_S(X))$. If $\Sigma\mathsf{K}(X)$ is a WD space, then X is a WD space.

By Lemma 5.1, Lemma 7.15 and Proposition 8.15, we get the following two corollaries.

Corollary 8.16 If X is a locally compact T_0 space and $\Sigma\mathsf{K}(X)$ is a WD space, then X is a WD space.

Corollary 8.17 Suppose that X is a T_0 space for which the Smyth power space $P_S(X)$ is first-countable. If $\Sigma\mathsf{K}(X)$ is a WD space, then X is a WD space.

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Thanks for your attention!